

Strong solutions to the Cauchy problem of the two-dimensional compressible Navier-Stokes-Smoluchowski equations with vacuum*

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Abstract. This paper studies the local existence of strong solutions to the Cauchy problem of the 2D fluid-particle interaction model with vacuum as far field density. Notice that the technique used by Ding et al. [14] for the corresponding 3D local well-posedness of strong solutions fails treating the 2D case, because the L^p -norm ($p > 2$) of the velocity u cannot be controlled in terms only of $\sqrt{\rho}u$ and ∇u here. In the present paper, we will use the framework of weighted approximation estimates introduced in [J. Li, Z. Liang, On classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum, J. Math. Pures Appl. (2014) 640–671] for Navier-Stokes equations to obtain the local existence of strong solutions provided the initial density and density of particles in the mixture do not decay very slowly at infinity. In particular, the initial density can have a compact support. This paper extends Fang et al.’s result [16] and Ding et al.’s result [14], in which, the existence is obtained when the space dimension $N = 1$ and $N = 3$ respectively.

Keywords: strong solutions, Cauchy problem, compressible Navier-Stokes-Smoluchowski equations, vacuum, two-dimensional space

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1 Introduction

In this paper, we consider a fluid-particle interaction model called as Navier-Stokes-Smoluchowski equations in [3, 8, 9], which in the whole spatial domain \mathbb{R}^2 as follows

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(p_F + \eta) = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u - (\eta + \beta \rho) \nabla \Phi, \\ \eta_t + \nabla \cdot (\eta(u - \nabla \Phi)) = \Delta \eta, \end{cases} \quad (1.1)$$

in $\mathbb{R}^2 \times \mathbb{R}^+$, with the far-field behavior

$$(\rho, u, \eta)(x, t) \rightarrow (0, 0, 0) \quad \text{as } |x| \rightarrow \infty, \quad t > 0, \quad (1.2)$$

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and initial data

$$\rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0, \quad \eta(x, 0) = \eta_0, \quad x \in \mathbb{R}^2. \quad (1.3)$$

Here $\rho : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^+$ is the density of the fluid, $u : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^2$ the velocity field, and the density of the particles in the mixture $\eta : (0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is related to the probability distribution function $f(t, x, \xi)$ in the macroscopic description through the relation

$$\eta(t, x) = \int_{\mathbb{R}^2} f(t, x, \xi) d\xi.$$

We also denote by p_F the pressure of the fluid, given by

$$p_F = p_F(\rho) = a\rho^\gamma, a > 0, \gamma > 1, \quad (1.4)$$

and the time independent external potential $\Phi = \Phi(x) : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is the effects of gravity and buoyancy, β is a constant reflecting the differences in how the external force affects the fluid and the particles, λ and μ are constant viscosity coefficients satisfying the physical condition:

$$\mu > 0, \quad \lambda + \mu \geq 0. \quad (1.5)$$

The fluid-particle interaction model arises in a lot of industrial procedures such as the analysis of sedimentation phenomenon which finds its applications in biotechnology, medicine, chemical engineering, and mineral processes. Such interaction systems are also used in combustion theory, when modeling diesel engines or rocket propulsors, see [6, 7, 24, 25]. The system consists in a Vlasov-Fokker-Planck equation to describe the microscopic motion of the particles coupled to the equations for the fluid. Generally speaking, at the microscopic scale, the cloud of particles is described by its distribution function $f(t, x, \xi)$, solution to a Vlasov-Fokker-Planck equation. The fluid, on the other hand, is modeled by macroscopic quantities, namely its density $\rho(x, t) \geq 0$ and its velocity field $u(x, t)$ (see [8]). If the fluid is compressible and isentropic, then (ρ, u) solves the compressible Euler (inviscid case) or Navier-Stokes system (viscous case) of equations. With the dynamic viscosity terms taken into consideration, system (1.1) was derived formally by Carrillo and Goudon [9]. They obtained the global existence and asymptotic behavior of the weak solutions to (1.1) following the framework of Lions [22] and Feireisl et al. [17, 18]. Without the dynamic viscosity terms in (1.1)₂, Carrillo and Goudon [8] gave the flowing regime and the bubbling regime under the two different scaling assumptions and investigated the stability and asymptotic limits finally. In dimension one, Fang et al. [16] proved the global existence and uniqueness of the classical large solution with vacuum. In dimension three, Ballew obtained the local in time existence of strong solutions in a bounded domain with the no-flux condition for the particle density in [3, 4] and studied Low Mach Number Limits under the confinement hypotheses for the spatial domain and external potential Φ in [5]. Recently, motivated by Kim et al. [10–12] on the Navier-Stokes equations, Ding et al. [14] obtained the local classical solutions of system (1.1) with vacuum in \mathbb{R}^3 .

When the density of the fluid $\eta = 0$, the system (1.1) becomes Navier-Stokes equations for the isentropic compressible fluids. Kim et al. proved some local existence results on strong solutions in a domain of \mathbb{R}^3 in [10, 11] and the radially symmetric solutions in an annular domain in [13]. Ding et al. [15] obtained global classical solutions with large initial data with vacuum in a bounded domain or exterior domain Ω of \mathbb{R}^n ($n \geq 2$). In a bounded or unbounded domain of \mathbb{R}^3 , Cho and Kim also got the local classical solutions [12], in which the initial density needs

not be bounded below away from zero. For the case that the initial density is allowed to vanish, Huang et al. [19] obtained the global existence of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three spatial dimensions with smooth initial data provided that the initial energy is suitably small. Recently, assumed that the initial density do not decay very slowly at infinity, Li and Liang [21] have obtained the local existence of the classical solutions to the two-dimensional Cauchy problem. After that, Li and Xin [20] extended the result of Li and Liang [21] to the global ones, and also get some decay estimates of solutions.

The aim of this paper is to establish the local existence of strong solutions to the Cauchy problem (1.1) in dimension two. Notice that the local well-posedness of strong solutions for dimension three case established by Ding et al. [14] is not admitted for the case of dimension two. This is mainly due to that in dimension two we fail to control the L^p -norm ($p > 2$) of the velocity u in terms only of $\sqrt{\rho}u$ and ∇u . Moreover, the coupling of u, η and Φ , and the presence of $\nabla \cdot (\eta u - \eta \nabla \Phi)$ bring additional difficulties. So, some new ideas and careful estimates are necessary to deal with the two dimension case. In the present paper, we will use the framework of weighted approximation estimates introduced in [21] for Navier-Stokes equations to overcome these difficulties.

Definition 1.1 *If all derivatives involved in (1.1) for (ρ, u, η) are regular distributions, and equations (1.1) hold almost everywhere in $\mathbb{R}^2 \times (0, T)$, then (ρ, u, η) is called a strong solution to (1.1).*

In this section, for $1 \leq r \leq \infty$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{s,r} = W^{s,r}(\mathbb{R}^2), \quad H^s = W^{s,2}. \quad (1.6)$$

Denote

$$\bar{x} \triangleq (e + |x|^2)^{1/2} \log^{1+\sigma_0}(e + |x|^2),$$

with $\sigma_0 > 0$, $B_N \triangleq \{x \in \mathbb{R}^2 \mid |x| < N\}$. The main result of this paper is stated as the following theorem:

Theorem 1.1 *Suppose that the initial data (ρ_0, u_0, η_0) satisfy*

$$\begin{aligned} \rho_0 &\geq 0, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \\ \nabla \eta_0 &\in L^2, \quad \bar{x}^{\frac{a}{2}} \eta_0 \in L^2, \quad \Phi \in H^4, \quad \sqrt{\rho_0} u_0 \in L^2, \end{aligned}$$

with $q > 2$ and $a > 1$. Then there exist $T_0, N > 0$ such that the problem (1.1)–(1.3) has a unique strong solution (ρ, u, η) on $\mathbb{R}^2 \times (0, T_0]$ satisfying

$$\left\{ \begin{array}{l} \rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \quad \bar{x}^a \rho \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho}u, \nabla u, \bar{x}^{-1}u, \sqrt{t}\sqrt{\rho}u_t \in L^\infty(0, T_0; L^2), \\ \nabla u \in L^2(0, T_0; H^1) \cap L^{\frac{q+1}{q}}(0, T_0; W^{1,q}), \sqrt{t}\nabla u \in L^2(0, T_0; W^{1,q}), \\ \eta, \nabla \eta, \bar{x}^{\frac{a}{2}}\eta, \sqrt{t}\eta_t \in L^\infty(0, T_0; L^2), \\ \nabla \eta \in L^2(0, T_0; H^1), \sqrt{t}\nabla u \in L^2(0, T_0; W^{1,q}), \\ \sqrt{\rho}u_t, \bar{x}^{\frac{a}{2}}\nabla \eta, \sqrt{t}\nabla u_t, \sqrt{t}\nabla \eta_t, \sqrt{t}\bar{x}^{-1}u_t \in L^2(\mathbb{R}^2 \times (0, T_0)), \end{array} \right. \quad (1.7)$$

and

$$\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x, t) dx. \quad (1.8)$$

The rest of the paper is organized as follows: In Section 2, we recall some elementary facts and inequalities used in the sequel. Sections 3 deals with an approximation problem (2.2) on B_R to derive uniform estimates for the unique strong solution with respect to R . Finally, the proof of Theorem 1.1 will be given in Section 4.

2 Preliminaries

Firstly, the follow local existence theory on bounded ball $B_R \triangleq \{x \in \mathbb{R}^2 : |x| < R\}$, where the initial density is strictly away from vacuum, can be shown by arguments as in [14].

Lemma 2.1 *For any given $R > 0$ and $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, assume that (ρ_0, u_0, η_0) satisfies*

$$(\rho_0, u_0, \eta_0) \in H^3(B_R), \quad \Phi \in H^4(B_R), \quad \inf_{x \in B_R} \rho_0 > 0. \quad (2.1)$$

Then there exist a small time $T_R > 0$ and a unique classical solution (ρ, u, η) to the following initial-boundary-value problem

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(p_F + \eta) = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u - (\eta + \beta \rho) \nabla \Phi - R^{-1} u, \\ \eta_t + \nabla \cdot (\eta(u - \nabla \Phi)) = \Delta \eta, \\ u = 0, \quad (\nabla \eta + \eta \nabla \Phi) \cdot n = 0, \quad x \in \partial B_R, \quad t > 0, \\ (\rho, u, \eta)(x, 0) = (\rho_0, u_0, \eta_0)(x), \quad x \in B_R, \end{cases} \quad (2.2)$$

on $B_R \times (0, T_R]$ such that

$$\begin{cases} \rho \in C([0, T_R]; H^3), \quad \rho_t \in L^\infty(0, T_R; H^2), \quad \sqrt{\rho} u_t \in L^\infty(0, T_R; L^2), \\ (u, \eta) \in C([0, T_R]; H^3) \cap L^2(0, T_R; H^4), \quad (u_t, \eta_t) \in L^\infty(0, T_R; H^1) \cap L^2(0, T_R; H^2), \\ (\sqrt{t} u, \sqrt{t} \eta) \in L^\infty(0, T_R; H^4), \quad (\sqrt{t} u_t, \sqrt{t} \eta_t) \in L^\infty(0, T_R; H^2), \\ (\sqrt{t} u_{tt}, \sqrt{t} \eta_{tt}) \in L^2(0, T_R; H^1), \quad \sqrt{t} \sqrt{\rho} u_{tt} \in L^2(0, T_R; L^2), \\ (tu_t, t\eta_t) \in L^\infty(0, T_R; H^3), \quad (tu_{tt}, t\eta_{tt}) \in L^\infty(0, T_R; H^1) \cap L^2(0, T_R; H^2), \\ t \sqrt{\rho} u_{ttt} \in L^\infty(0, T_R; L^2), \quad (t^{\frac{3}{2}} u_{tt}, t^{\frac{3}{2}} \eta_{tt}) \in L^\infty(0, T_R; H^2), \\ t^{\frac{3}{2}} \sqrt{\rho} u_{ttt} \in L^\infty(0, T_R; L^2), \quad (t^{\frac{3}{2}} u_{ttt}, t^{\frac{3}{2}} \eta_{ttt}) \in L^2(0, T_R; H^1), \end{cases} \quad (2.3)$$

where we denote $L^2 = L^2(B_R)$ and $H^k = H^k(B_R)$ for positive integer k .

Next, for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$, the following weighted L^p -bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) \triangleq \{v \in H_{\text{loc}}^1(\Omega) : |\nabla v| \in L^2(\Omega)\}$ can be found in [23, theorem B.1].

Lemma 2.2 *For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant C such that for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$ and for any $v \in \tilde{D}^{1,2}(\Omega)$,*

$$\left(\int_{\Omega} \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} dx \right)^{\frac{1}{m}} \leq C \|v\|_{L^2(B_1)} + C \|\nabla v\|_{L^2(\Omega)}. \quad (2.4)$$

Lemma 2.3 (Lemma 2.4 in [21]) *Let \bar{x} and σ_0 be as in Theorem 1.1 with $\Omega = \mathbb{R}^2$ or $\Omega = B_R$, and $\rho \in L^1(\Omega) \cap L^\gamma(\Omega)$ with $\gamma > 1$ be a non-negative function satisfying*

$$\int_{B_{N_1}} \rho dx \geq M_1, \quad \int_{\Omega} \rho^\gamma dx \leq M_2,$$

with $M_1, M_2 > 0$, and $B_{N_1} \subset \Omega$ ($N_1 \geq 1$). Then for every $v \in \tilde{D}^{1,2}(\Omega)$, there is $C = C(M_1, M_2, N_1, \gamma, \sigma_0) > 0$ such that

$$\|v\bar{x}^{-1}\|_{L^2(\Omega)} \leq C\|\sqrt{\rho}v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)}. \quad (2.5)$$

Moreover, for $\varepsilon > 0$ and $\sigma > 0$ there is $C = C(\varepsilon, \eta, M_1, M_2, N_1, \gamma, \sigma_0) > 0$ such that every $v \in \tilde{D}^{1,2}(\Omega)$ satisfies

$$\|v\bar{x}^{-\sigma}\|_{L^{\frac{2+\varepsilon}{\sigma}}(\Omega)} \leq C\|\sqrt{\rho}v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)}, \quad (2.6)$$

with $\tilde{\sigma} = \min\{1, \sigma\}$.

Next, the following L^p -bound for elliptic systems, whose proof is similar to that of [10, lemma 12], is a direct consequence of the combination of a well-known elliptic theory due to Agmon-Douglis-Nirenberg [1, 2] with a standard scaling procedure.

Lemma 2.4 *For $p > 1$ and $k \geq 0$, there exists a positive constant C depending only on p and k such that*

$$\|\nabla^{k+2}v\|_{L^p(B_R)} \leq C\|\Delta v\|_{W^{1,p}(B_R)}, \quad (2.7)$$

for every $v \in W^{k+2,p}(B_R)$ satisfying either

$$v \cdot n = 0, \quad \operatorname{rot} v = 0, \quad \text{on } \partial B_R,$$

or

$$v = 0, \quad \text{on } \partial B_R.$$

3 A priori estimates for approximation problem

Throughout this section and the next, for $p \in [1, \infty]$ and $k \geq 0$, we denote

$$\int f dx = \int_{B_R} f dx, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^k = W^{k,2}.$$

Moreover, for $R > 4N_0 \geq 4$, assume that (ρ_0, u_0, η_0) satisfies, in addition to (2.1), that

$$\frac{1}{2} \leq \int_{B_{N_0}} \rho_0(x) dx \leq \int_{B_R} \rho_0(x) dx \leq \frac{3}{2}. \quad (3.1)$$

Lemma 2.1 thus yields that there exists some $T_R > 0$ such that the initial-boundary value problem (2.2) has a unique classical solution (ρ, u, η) on $B_R \times [0, T_R]$ satisfying (2.3).

For \bar{x}, σ_0, a and q as in theorem 1.1, the main aim of this section is to derive the following key a priori estimate on ψ defined by

$$\psi(t) \triangleq 1 + \|\sqrt{\rho}u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \eta\|_{L^2} + \|\bar{x}^{\frac{a}{2}}\eta\|_{L^2} + \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} + R^{-1}\|u\|_{L^2}. \quad (3.2)$$

Proposition 3.1 Assume that (ρ_0, u_0, η_0) satisfies (2.1) and (3.1). Then there exist $T_0, M > 0$, both depending only on $\mu, \gamma, q, a, \eta_0, N_0$, and E_0 , such that

$$\sup_{0 \leq t \leq T_0} \psi(t) + \int_0^{T_0} (\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla^2 u\|_{L^q}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \eta\|_{L^2}^2) dt \leq M. \quad (3.3)$$

where

$$E_0 \triangleq \|\sqrt{\rho_0} u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + \|\nabla \eta_0\|_{L^2} + \|\bar{x}^a \rho_0\|_{L^1 \cap H^1 \cap W^{1,q}} + \|\bar{x}^{\frac{a}{2}} \eta_0\|_{L^2},$$

To prove proposition 3.1, whose proof will be postponed to the end of this section, we begin with the following standard energy estimate for (ρ, u, η) .

Lemma 3.1 Let (ρ, u, η) be a smooth solution to the initial-boundary value problem (2.2). Then there exists $T_1 = T_1(N_0, E_0) > 0$ such that for all $t \in (0, T_1]$

$$\begin{aligned} \sup_{0 \leq s \leq t} \int \left[\frac{1}{2} \rho |u|^2 + \frac{a}{\gamma-1} \rho^\gamma + \eta \ln \eta + (\beta \rho + \eta) \Phi \right] dx \\ + \int_0^t \int \left[|\nabla u|^2 + |2\nabla \sqrt{\eta} + \sqrt{\eta} \nabla \Phi|^2 \right] dx ds \leq C, \end{aligned} \quad (3.4)$$

and moreover,

$$\sup_{0 \leq s \leq t} \int \eta^2 dx + \int_0^t \int |\nabla \eta|^2 dx ds \leq C. \quad (3.5)$$

where and throughout the paper, denote by C generic positive constants depending only on the fixed constants $\mu, \lambda, \gamma, \beta, a, q, \sigma_0, N_0, E_0$, and $\|\Phi\|_{H^4(\mathbb{R}^2)}$.

Proof. First, multiplying $(2.2)_2$ by u , integrating the resulting equation over B_R and using Eq.(2.2)₁, we have

$$\begin{aligned} \frac{d}{dt} \int \left[\frac{1}{2} \rho |u|^2 + \frac{a}{\gamma-1} \rho^\gamma \right] dx \\ + \int \left[\mu |\nabla u|^2 + (\mu + \lambda) |\operatorname{div} u|^2 + u \cdot \nabla \eta + (\beta \rho + \eta) \nabla \Phi \cdot u + R^{-1} |u|^2 \right] dx = 0, \end{aligned} \quad (3.6)$$

where we have used the fact

$$\int \rho^\gamma \nabla \cdot u dx = \int \rho^{\gamma-1} \rho \nabla \cdot u dx = - \int (\rho_t + u \cdot \nabla \rho) \rho^{\gamma-1} = - \frac{d}{dt} \int \frac{\rho^\gamma}{\gamma} dx + \int \frac{\rho^\gamma}{\gamma} \nabla \cdot u,$$

so that

$$-a \int \rho^\gamma \nabla \cdot u = \frac{d}{dt} \int \frac{a \rho^\gamma}{\gamma-1}.$$

Using $(2.2)_1$ and $(2.2)_3$, we have

$$\begin{aligned} \int (\eta + \beta \rho) \nabla \Phi \cdot u dx &= - \int \operatorname{div}(\eta u) dx - \int \beta \operatorname{div}(\rho u) \Phi dx \\ &= \frac{d}{dt} \int \beta \rho \phi dx + \int \left[\eta_t - \nabla \cdot (\eta \nabla \phi) - \Delta \eta \right] \Phi dx \end{aligned}$$

$$= \frac{d}{dt} \int (\eta + \beta\rho)\Phi dx + \int (\eta\nabla\Phi + \nabla\eta)\nabla\Phi dx. \quad (3.7)$$

Multiplying (2.2)₃ by $\log \eta$, integrating the resulting equation over B_R , and using the boundary condition (2.2)₄, one deduces that

$$\begin{aligned} & \int \eta_t \log \eta dx - \int \left[\eta u - \eta \nabla \Phi - \nabla \eta \right] \frac{\nabla \eta}{\eta} dx \\ &= \frac{d}{dt} \int \eta \log \eta dx - \int \left[u \cdot \nabla \eta - \nabla \Phi \cdot \nabla \eta - \frac{|\nabla \eta|^2}{\eta} \right] dx = 0. \end{aligned} \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6), we immediately complete the proof of (3.4).

Next, multiplying (2.2)₂ by η , integrating the resulting equation over B_R , using boundary condition (2.2)₄, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \eta^2 dx + \int |\nabla \eta|^2 dx &= \int \eta(u - \nabla \Phi) \nabla \eta dx \\ &= \int \eta u \nabla \eta dx - \int \eta \nabla \Phi \nabla \eta dx \\ &= - \int \operatorname{div} u \eta^2 dx - \int \eta \nabla \Phi \nabla \eta dx \\ &\leq \frac{1}{4} \int |\nabla \eta|^2 dx + C \int |\eta|^2 |\nabla u| dx + C \int \eta^2 |\nabla \Phi|^2 dx \\ &\leq \frac{1}{4} \int |\nabla \eta|^2 dx + C \int \eta^2 |\nabla \Phi|^2 dx + C \|\nabla u\|_{L^2} \|\eta\|_{L^4}^2 \\ &\leq \frac{1}{4} \int |\nabla \eta|^2 dx + C \int \eta^2 |\nabla \Phi|^2 dx + C \|\nabla u\|_{L^2} \|\eta\|_{L^2} \|\nabla \eta\|_{L^2} \\ &\leq \frac{1}{2} \int |\nabla \eta|^2 dx + C \int \eta^2 dx + C \|\nabla u\|_{L^2}^2 \|\eta\|_{L^2}^2. \end{aligned} \quad (3.9)$$

According to energy inequality (3.4), we have $\int_0^t \int |\nabla u|^2 dx ds \leq C$. Thus, we can use Gronwall's inequality to deduce that

$$\sup_{0 \leq s \leq t} \int \eta^2 dx + \int_0^t \int |\nabla \eta|^2 dx ds \leq C. \quad (3.10)$$

Lemma 3.2 *Under the conditions of Proposition 3.1, let (ρ, u, η) be a smooth solution to the initial-boundary value problem (2.1)-(2.2). Then there exists $T_1 = T_1(N_0, E_0) > 0$ and $\alpha = \alpha(\gamma, q) > 1$ such that for all $t \in (0, T_1]$*

$$\sup_{0 \leq s \leq t} \|\bar{x}^{\frac{\alpha}{2}} \eta\|_{L^2}^2 + \int_0^t \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^2 ds \leq C, \quad (3.11)$$

$$\sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2}^2 + \|\nabla \eta\|_{L^2}^2) + \int_0^t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\eta_t\|_{L^2}^2 + \|\Delta \eta\|_{L^2}^2) ds \leq C \int_0^t \psi^\alpha ds + C. \quad (3.12)$$

Proof. First, we always assume that $t \leq T_1$. The conservation of ρ with (2.2)₁ yields that there exists $T_1 > 0$ such that

$$\inf_{0 \leq t \leq T_1} \int_{B_{2N_0}} \rho dx \geq \frac{1}{4}, \quad (3.13)$$

that is (3.8) in [21]. Furthermore, corresponding to (3.10) obtained in [21], we have by (3.4), (3.13), and Lemma 2.3 that

$$\|\rho^\eta u\|_{L^{\frac{2+\varepsilon}{\sigma}}} + \|u\bar{x}^{-\eta}\|_{L^{\frac{2+\varepsilon}{\sigma}}} \leq C(\varepsilon, \sigma)\psi^{1+\sigma}, \quad t \in (0, T_1] \quad (3.14)$$

with $\tilde{\sigma} = \min\{1, \sigma\}$.

Next, to obtain (3.11), multiplying (2.2)₃ by $\eta\bar{x}^a$ and integrating by parts yield

$$\begin{aligned} & \frac{1}{2} \left(\int |\eta|^2 \bar{x}^a dx \right)_t + \int |\nabla \eta|^2 \bar{x}^a dx \\ &= \frac{1}{2} \int \eta^2 \Delta \bar{x}^a dx - \int \nabla \cdot (\eta u - \eta \nabla \Phi) \eta \bar{x}^a dx \\ &= \frac{1}{2} \int \eta^2 \Delta \bar{x}^a dx - \int \eta u \cdot \nabla \eta \bar{x}^a dx - \int \eta^2 (\nabla \cdot u) \bar{x}^a dx \\ & \quad + \int \eta \nabla \eta \cdot \nabla \Phi \bar{x}^a dx + \int \eta^2 \Delta \Phi \bar{x}^a dx \\ &\leq C \int |\eta|^2 \Delta \bar{x}^a dx + C \int |\eta|^2 |\nabla u| \bar{x}^a dx + C \int |\eta|^2 |u| \cdot \nabla \bar{x}^a dx \\ & \quad + C \int |\eta|^2 |\Delta \Phi| \bar{x}^a dx + C \int |\eta|^2 |\nabla \Phi| \nabla \bar{x}^a dx \\ &\triangleq \sum_{i=1}^5 I_i. \end{aligned} \quad (3.15)$$

Direct calculations yield that

$$I_1 \leq C \int |\eta|^2 \bar{x}^a \bar{x}^{-2} \log^{2(1+\sigma_0)}(e + |x|^2) dx \leq C \int |\eta|^2 \bar{x}^a dx, \quad (3.16)$$

$$\begin{aligned} I_2 &\leq C \int |\nabla u| |\eta|^2 \bar{x}^a dx \\ &\leq C \|\nabla u\|_{L^2} \|\eta \bar{x}^a\|_{L^4}^2 \\ &\leq C \|\nabla u\|_{L^2} \|\eta \bar{x}^a\|_{L^2} (\|\nabla \eta \bar{x}^{\frac{a}{2}}\|_{L^2} + \|\eta \nabla \bar{x}^{\frac{a}{2}}\|_{L^2}) \\ &\leq C (\|\nabla u\|_{L^2}^2 + 1) \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{1}{4} \|\nabla \eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2, \end{aligned} \quad (3.17)$$

$$\begin{aligned} I_3 &\leq C \int \bar{x}^a |\eta|^2 \bar{x}^{-\frac{3}{4}} |u| \bar{x}^{-\frac{1}{4}} \log^{1+\sigma_0}(e + |x|^2) dx \\ &\leq C \|\eta \bar{x}^{\frac{a}{2}}\|_{L^4} \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2} \|u \bar{x}^{-\frac{3}{4}}\|_{L^4} \\ &\leq C \|\eta \bar{x}^{\frac{a}{2}}\|_{L^4}^2 + C \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2 (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\ &\leq C (1 + \|\nabla u\|_{L^2}^2) \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + \frac{1}{4} \|\nabla \eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2, \end{aligned} \quad (3.18)$$

$$\begin{aligned} I_4 + I_5 &\leq C \int |\eta|^2 \bar{x}^a dx + C \int |\eta|^2 \bar{x}^a \bar{x}^{-1} \log^{1+\sigma_0}(e + |x|^2) dx \\ &\leq C \int |\eta|^2 \bar{x}^a dx. \end{aligned} \quad (3.19)$$

Putting (3.17)-(3.19) into (3.15), after using Gronwall's inequality and (3.4), we have

$$\sup_{0 \leq s \leq t} \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2}^2 + \int_0^t \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 dx \leq C \exp \left\{ C \int_0^t (1 + \|\nabla u\|_{L^2}^2) ds \right\} \leq C. \quad (3.20)$$

Next, to prove (3.12), multiplying Eqs. (2.2)₂ by u_t and integration by parts yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left[(2\mu + \lambda)(\operatorname{div} u)^2 + \mu\omega^2 + R^{-1}|u|^2 \right] dx + \int \rho|u_t|^2 dx \\ & \leq C \int \rho|u|^2 |\nabla u|^2 dx + 2 \int (p_F + \eta) \operatorname{div} u_t dx - \int (\beta\rho + \eta) \nabla \Phi \cdot u_t dx, \end{aligned} \quad (3.21)$$

where $\omega \triangleq \operatorname{rot} u$ is defined in the following (3.40).

We estimate each term on the right-hand side of (3.21) as follows:

First, the Gagliardo-Nirenberg inequality implies that for all $p \in (2, +\infty)$,

$$\begin{aligned} \|\nabla u\|_{L^p} & \leq C(p) \|\nabla u\|_{L^2}^{2/p} \|\nabla u\|_{H^1}^{1-2/p} \\ & \leq C(p) \psi + C(p) \psi \|\nabla^2 u\|_{L^2}^{1-2/p}, \end{aligned} \quad (3.22)$$

which together with (3.14) yields that for $\sigma > 0$ and $\tilde{\sigma} = \min\{1, \sigma\}$,

$$\begin{aligned} \int \rho^\sigma |u|^2 |\nabla u|^2 dx & \leq C \|\rho^{\sigma/2} u\|_{L^{8/\tilde{\sigma}}}^2 \|\nabla u\|_{L^{8/(4-\tilde{\sigma})}}^2 \\ & \leq C(\sigma) \psi^{4+2\sigma} (1 + \|\nabla^2 u\|_{L^2}^{\tilde{\sigma}/2}) \\ & \leq C(\varepsilon, \sigma) \psi^{\alpha(\sigma)} + \varepsilon \psi^{-2} \|\nabla^2 u\|_{L^2}^2. \end{aligned} \quad (3.23)$$

Next, noticing that p_F satisfies

$$p_{Ft} + \operatorname{div}(p_F u) + (\gamma - 1)p_F \operatorname{div} u = 0. \quad (3.24)$$

we deduce from (2.2)₁ and the Sobolev inequality that

$$\begin{aligned} 2 \int p_F \operatorname{div} u_t dx & = 2 \frac{d}{dt} \int p_F \operatorname{div} u dx - 2 \int p_F u \cdot \nabla \operatorname{div} u dx + 2(\gamma - 1) \int p_F (\operatorname{div} u)^2 dx \\ & \leq 2 \frac{d}{dt} \int p_F \operatorname{div} u dx + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon) \psi^\alpha. \end{aligned} \quad (3.25)$$

Moreover, we have

$$\begin{aligned} & - \int (\nabla \eta + \eta \nabla \Phi) \cdot u_t dx - \int \beta \rho \nabla \Phi \cdot u_t dx \\ & = \frac{d}{dt} \int (\nabla \eta + \eta \nabla \Phi) \cdot u dx - \int (\nabla \eta + \eta \nabla \Phi)_t \cdot u dx - \int \beta \rho \nabla \Phi \cdot u_t dx \\ & = \frac{d}{dt} \int (\eta \nabla \cdot u - \eta \nabla \Phi \cdot u) dx - \int \eta_t \nabla \cdot u dx + \int \eta_t \nabla \Phi \cdot u dx - \int \beta \rho \nabla \Phi \cdot u_t dx \\ & = \frac{d}{dt} \int (\eta \nabla \cdot u - \eta \nabla \Phi \cdot u) dx - \int \eta_t \nabla \cdot u dx \\ & \quad + \int (\Delta \eta - \nabla \cdot (\eta u - \eta \nabla \Phi)) \nabla \Phi \cdot u dx - \int \beta \rho \nabla \Phi \cdot u_t dx \\ & = \frac{d}{dt} \int (\eta \nabla \cdot u - \eta \nabla \Phi \cdot u) dx - \int \eta_t \nabla \cdot u dx \\ & \quad - \int \nabla u \cdot \nabla \eta \cdot \nabla \Phi dx - \int u \cdot \nabla \eta \cdot \Delta \Phi dx + \int \eta u \cdot \nabla u \cdot \nabla \Phi dx - \int \beta \rho \nabla \Phi \cdot u_t dx \end{aligned}$$

$$\begin{aligned}
& + \int \eta u^2 \cdot \Delta \Phi dx - \int \eta u \cdot \nabla \Phi \cdot \Delta \Phi dx - \int \eta \nabla u |\nabla \Phi|^2 dx, \\
& \triangleq \frac{d}{dt} J_0 + \sum_{i=1}^8 J_i.
\end{aligned} \tag{3.26}$$

Direct calculations yield that

$$J_1 \leq C \int |\eta_t| |\nabla u| dx \leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + C \int |\nabla u|^2 dx, \tag{3.27}$$

$$J_2 \leq C \|\nabla \Phi\|_{L^\infty} \int |\nabla u| |\nabla \eta| dx \leq C \|\nabla \eta\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2, \tag{3.28}$$

$$\begin{aligned}
J_3 & \leq C \int |u| \bar{x}^{-\frac{a}{2}} |\nabla \eta| \bar{x}^{\frac{a}{2}} \Delta \Phi dx \\
& \leq C \|\Delta \Phi\|_{L^4} \|u \bar{x}^{-\frac{a}{2}}\|_{L^4} \|\nabla \eta \bar{x}^{\frac{a}{2}}\|_{L^2} \\
& \leq C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2),
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
J_4 & \leq C \int |\eta| \bar{x}^{\frac{a}{2}} |u| \bar{x}^{-\frac{a}{2}} |\nabla u| |\nabla \Phi| dx \\
& \leq C \|\nabla \Phi\|_{L^\infty} \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2} \|u \bar{x}^{-\frac{a}{2}}\|_{L^4} \|\nabla u\|_{L^4} \\
& \leq C \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\
& \leq C \psi^\alpha + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
J_5 & \leq C \|\nabla \Phi\|_{L^\infty} \int \sqrt{\rho} \sqrt{\rho} u_t dx \\
& \leq \frac{1}{2} \int \rho |u_t|^2 dx + C \|\nabla \Phi\|_{L^\infty} \int \rho_0 dx \\
& \leq \frac{1}{2} \int \rho |u_t|^2 dx + C,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
J_6 & \leq C \int |\eta| \bar{x}^{\frac{a}{2}} |u|^2 \bar{x}^{-\frac{a}{2}} |\Delta \Phi| dx \\
& \leq C \|\Delta \Phi\|_{L^\infty} \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2} \|u \bar{x}^{-\frac{a}{4}}\|_{L^4}^2 \\
& \leq C \|\Delta \Phi\|_{L^\infty} \|\eta \bar{x}^{\frac{a}{2}}\|_{L^2} (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
& \leq C \psi^\alpha,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
J_7 & \leq C \|\Delta \Phi\|_{L^\infty} \int |\eta| \bar{x}^{\frac{a}{2}} |u| \bar{x}^{-\frac{a}{2}} |\nabla \Phi| dx \\
& \leq C \|\eta \bar{x}^{\frac{a}{2}}\|_{L^4} \|u \bar{x}^{-\frac{a}{2}}\|_{L^4} \|\nabla \Phi\|_{L^2} \\
& \leq C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + C \psi^\alpha,
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
J_8 & \leq C \|\nabla \Phi\|_{L^\infty}^2 \int |\eta| |\nabla u| dx \\
& \leq C \|\nabla \Phi\|_{L^\infty}^2 \|\eta\|_{L^2} \|\nabla u\|_{L^2} \\
& \leq C \psi^\alpha + C.
\end{aligned} \tag{3.34}$$

Substituting (3.27)-(3.40) into (3.26), and combining (3.26) and (3.21) lead to

$$\frac{1}{2} \frac{d}{dt} \int \left[(2\mu + \lambda) (\operatorname{div} u)^2 + \mu \omega^2 + R^{-1} |u|^2 \right] dx + \int \rho |u_t|^2 dx$$

$$\leq \frac{d}{dt}B(t) + \varepsilon\psi^{-1}\|\nabla^2 u\|_{L^2}^2 + C\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + C\psi^\alpha, \quad (3.35)$$

where

$$\begin{aligned} B(t) &= -2 \int p_F \operatorname{div} u dx + \int (\eta \nabla \cdot u - \eta \nabla \Phi u) dx \\ &\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C \|p_F\|_{L^2}^2 + C \|\eta\|_{L^2}^2 + C \|\nabla \Phi\|_{L^4} \|\bar{x}^{\frac{\alpha}{2}} \eta\|_{L^2} \|u \bar{x}^{-\frac{\alpha}{2}}\|_{L^4} \\ &\leq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + C \|p_F\|_{L^2}^2 + C, \end{aligned} \quad (3.36)$$

owing to (3.4), (3.5), (3.20) and (3.14).

Moreover, multiplying the equation (2.2)₃ by η_t and integrating the result equation with respect to x over B_R , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \|\eta_t\|_{L^2}^2 \\ &= \int \nabla \cdot (\eta(u - \nabla \Phi)) \cdot \eta_t dx \\ &\leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + C \int |u|^2 |\nabla \eta|^2 dx + C \int |\eta|^2 |\operatorname{div} u|^2 dx + C \int |\nabla \eta|^2 |\nabla \Phi|^2 dx + C \int |\eta|^2 |\Delta \Phi|^2 dx \\ &\leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + C \int |u|^2 \bar{x}^{-\frac{\alpha}{2}} |\nabla \eta| \bar{x}^{\frac{\alpha}{2}} |\nabla \eta| dx + C \|\eta\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + C \int |\nabla \eta|^2 dx + C \\ &\leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + C \|\bar{x}^{-\frac{\alpha}{4}} u\|_{L^8}^2 \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2} \|\nabla \eta\|_{L^4} + C \|\eta\|_{L^2} \|\nabla \eta\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} + C \int |\nabla \eta|^2 dx + C \\ &\leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^2 + C \psi^\alpha \|\nabla \eta\|_{L^4}^2 + C(\varepsilon) \psi^\alpha \\ &\leq \frac{1}{2} \|\eta_t\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \eta\|_{L^2}^2 + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^2 + C(\varepsilon) \psi^\alpha. \end{aligned} \quad (3.37)$$

From (2.2)₃, taking it by L^2 -norm, using Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} \|\Delta \eta\|_{L^2} &\leq C \|\eta_t\|_{L^2} + C \|\nabla \cdot (\eta u - \eta \nabla \Phi)\|_{L^2} \\ &\leq C \|\eta_t\|_{L^2} + C (\|u \nabla \eta\|_{L^2} + \|\eta \nabla u\|_{L^2} + \|\nabla \Phi\|_{L^\infty} \|\nabla \eta\|_{L^2} + \|\Delta \Phi\|_{L^\infty} \|\eta\|_{L^2}) \\ &\leq C \|\eta_t\|_{L^2} + C \|\bar{x}^{-\frac{\alpha}{4}} u\|_{L^8} \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^{\frac{1}{2}} \|\nabla \eta\|_{L^4}^{\frac{1}{2}} \\ &\quad + C \|\eta\|_{L^2}^{\frac{1}{2}} \|\nabla \eta\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} + C \|\nabla \eta\|_{L^2} + C \\ &\leq C \|\eta_t\|_{L^2} + \frac{1}{2} \|\nabla^2 \eta\|_{L^2}^2 + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^2 + C(\varepsilon) \psi^\alpha \end{aligned} \quad (3.38)$$

Finally, to estimate the last term on the right-hand side of (3.37), (3.38), and (3.35), denoting $\nabla^\perp \triangleq (\partial_2, -\partial_1)$, we rewrite the momentum equation (2.2)₂ as

$$R^{-1}u + \rho \dot{u} = \nabla F + \mu \nabla^\perp \omega - (\eta + \beta \rho) \nabla \Phi, \quad (3.39)$$

where

$$\dot{f} \triangleq f_t + u \cdot \nabla f, \quad F \triangleq (2\mu + \lambda) \operatorname{div} u - p_F - \eta, \quad \omega \triangleq \nabla^\perp \cdot u \quad (3.40)$$

are the material derivative of f , the effective viscous flux and the vorticity respectively. Thus, (3.39) implies that ω satisfies

$$\begin{cases} \mu \Delta \omega = \nabla^\perp (\rho \dot{u} + (\eta + \beta \rho) \nabla \Phi + R^{-1} u), & \text{in } B_R, \\ \omega = 0, & \text{on } \partial B_R. \end{cases} \quad (3.41)$$

Applying the standard L^p -estimate to (3.41) yields that, for $p \in (1, \infty)$,

$$\|\nabla \omega\|_{L^p} \leq C(\rho)(\|\rho \dot{u}\|_{L^p} + \|(\eta + \beta \rho) \nabla \Phi\|_{L^p} + R^{-1} \|u\|_{L^p}),$$

which together with (3.39) gives

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(\rho)(\|\rho \dot{u}\|_{L^p} + \|(\eta + \beta \rho) \nabla \Phi\|_{L^p} + R^{-1} \|u\|_{L^p}), \quad (3.42)$$

It follows from (2.7) and (3.42) that for $p \in [2, q]$,

$$\begin{aligned} \|\nabla^2 u\|_{L^p} &\leq C \|\nabla \omega\|_{L^p} + C \|\nabla \operatorname{div} u\|_{L^p} \\ &\leq C(\|\nabla \omega\|_{L^p} + \|(2\mu + \lambda) \nabla \operatorname{div} u\|_{L^p}) \\ &\leq C(\|\rho \dot{u}\|_{L^p} + \|(\eta + \beta \rho) \nabla \Phi\|_{L^p} + \|\nabla p_F\|_{L^p} + \|\nabla \eta\|_{L^p} + R^{-1} \|u\|_{L^p}), \end{aligned} \quad (3.43)$$

which together with (3.5), (3.22) and (3.23) leads to

$$\begin{aligned} \|\nabla^2 u\|_{L^2} &\leq C \sqrt{\psi} \|\sqrt{\rho} u_t\|_{L^2} + C \|\rho u \cdot \nabla u\|_{L^2} + C \psi^\alpha \\ &\leq C \sqrt{\psi} \|\sqrt{\rho} u_t\|_{L^2} + C \psi^\alpha + \frac{1}{2} \|\nabla^2 u\|_{L^2}. \end{aligned} \quad (3.44)$$

Putting (3.44) into (3.37), (3.38), and (3.35), integrating the resulting inequality over $(0, t)$ and choosing ε suitably small yield

$$\begin{aligned} &R^{-1} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \eta\|_{L^2}^2 + \int_0^t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\eta_t\|_{L^2}^2 + \|\nabla^2 \eta\|_{L^2}^2) ds \\ &\leq C + C \|p_F\|_{L^2}^2 + C \int_0^t \psi^\alpha ds + C \int_0^t \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 ds \\ &\leq C + C \|p_F\|_{L^2}^2 + C \int_0^t \psi^\alpha ds \\ &\leq C + C \int_0^t \psi^\alpha ds, \end{aligned} \quad (3.45)$$

where we have used (3.15) and the following estimate

$$\|p_F\|_{L^2}^2 \leq \|p_F(\rho_0)\|_{L^2}^2 + C \int_0^t \|p_F\|_{L^1}^{1/2} \|p_F\|_{L^\infty}^{3/2} \|\nabla u\|_{L^2} ds \leq C + C \int_0^t \psi^\alpha ds \quad (3.46)$$

due to (3.24). The proof of lemma 3.2 is completed.

Lemma 3.3 *Let (ρ, u, η) and T_1 be as in Lemma 3.2. Then, for all $t \in (0, T_1]$,*

$$\sup_{0 \leq s \leq t} s \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + \int_0^t s \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2 ds \leq C \exp \left\{ \int_0^t \psi^\alpha ds \right\}, \quad (3.47)$$

$$\sup_{0 \leq s \leq t} s \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^t s (\|\nabla u_t\|_{L^2}^2 + R^{-1} |u_t|^2) ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \quad (3.48)$$

Proof. Differentiating (2.2)₂ with respect to t gives

$$\begin{aligned} & \rho u_{tt} + \rho u \cdot \nabla u_t - \mu \nabla^\perp \omega_t - \nabla((2\mu + \lambda) \operatorname{div} u_t) + R^{-1} u_t \\ &= -\rho_t(u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla(p_{Ft} + \eta_t) - (\eta_t + \beta \rho_t) \nabla \Phi. \end{aligned} \quad (3.49)$$

Multiplying (3.49) by u_t and integrating the resulting equation over B_R , we obtain after using (2.2)₁ that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int ((2\mu + \lambda) (\operatorname{div} u_t)^2 + \mu \omega_t^2 + R^{-1} |u_t|^2) dx \\ &= -2 \int \rho u \cdot \nabla u_t \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \\ & \quad - \int \rho u_t \cdot \nabla u \cdot u_t dx + \int (p_{Ft} + \eta_t) \operatorname{div} u_t dx - \int (\eta_t + \beta \rho_t) \nabla \Phi \cdot u_t dx \\ &\triangleq \Psi(t) + \int \eta_t \operatorname{div} u_t dx - \int (\eta_t + \beta \rho_t) \nabla \Phi \cdot u_t dx. \end{aligned} \quad (3.50)$$

By the arguments (3.27)–(3.31) for the proof of Lemma 3.3 in [21], it follows from (3.4), (3.5) and (3.14) for $\varepsilon \in (0, 1)$ that

$$\Psi(t) \leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \psi^\alpha (\|\nabla^2 u\|_{L^2}^2 + \|\rho^{1/2} u_t\|_{L^2}^2 + 1). \quad (3.51)$$

On the other hand,

$$\begin{aligned} & \int \eta_t \operatorname{div} u_t dx - \int (\eta_t + \beta \rho_t) \nabla \Phi u_t dx \\ &= \int (\Delta \eta - \nabla \cdot (\eta(u - \nabla \Phi))) \cdot \operatorname{div} u_t dx - \int (\Delta \eta - \nabla \cdot (\eta(u - \nabla \Phi)) - \beta \operatorname{div}(\rho u)) \nabla \Phi \cdot u_t dx \\ &\leq C \int |\Delta \eta| |\nabla u_t| dx + C \int |u| |\nabla \eta| |\nabla u_t| dx + C \int \eta |\nabla u| |\nabla u_t| dx \\ & \quad + C \int |\nabla \eta| |\nabla u_t| |\nabla \Phi| dx + C \int \eta |\nabla u_t| |\Delta \Phi| dx + C \int |\nabla \eta| |\Delta \Phi| |u_t| dx \\ & \quad + C \int \eta |u| |\Delta \Phi| |u_t| dx + C \int \eta |u| |\nabla \Phi| |\nabla u_t| dx + C \int \eta |\nabla \Phi| |\Delta \Phi| |u_t| dx \\ & \quad + C \int \eta |\nabla \Phi|^2 |\nabla u_t| dx + C \int |\nabla \rho| |u| |\nabla \Phi| |u_t| dx + C \int \rho |\nabla u| |\nabla \Phi| |u_t| dx \\ &\triangleq \sum_{i=1}^{12} R_i. \end{aligned} \quad (3.52)$$

Using Gagliardo-Nirenberg and Hölder's inequalities, we get

$$\begin{aligned} R_1 &\leq C \|\Delta \eta\|_{L^2} \|\nabla u_t\|_{L^2} \\ &\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \|\eta_t\|_{L^2}^2 + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^2}^2 + C(\varepsilon) \psi^\alpha, \\ R_2 &\leq C \int |u| \bar{x}^{-\frac{\alpha}{2}} |\nabla \eta| \bar{x}^{\frac{\alpha}{2}} |\nabla u_t| dx \\ &\leq C \|u \bar{x}^{-\frac{\alpha}{2}}\|_{L^4} \|\bar{x}^{\frac{\alpha}{2}} \nabla \eta\|_{L^4} \|\nabla u_t\|_{L^2} \end{aligned} \quad (3.53)$$

$$\begin{aligned}
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \|u \bar{x}^{-\frac{a}{2}}\|_{L^4}^2 \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2} \|\bar{x}^{\frac{a}{2}} \nabla^2 \eta\|_{L^2} \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + \frac{1}{8} \|\bar{x}^{\frac{a}{2}} \nabla^2 \eta\|_{L^2}^2 + C \psi^\alpha \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2,
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
R_3 &\leq C \|\nabla u_t\|_{L^2} \|\eta\|_{L^4} \|\nabla u\|_{L^4} \\
&\leq C \|\nabla u_t\|_{L^2} \|\nabla \eta\|_{L^2}^{\frac{1}{2}} \|\eta\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon) \psi^\alpha,
\end{aligned} \tag{3.55}$$

$$R_4 \leq C \|\nabla \Phi\|_{L^\infty} \|\nabla \eta\|_{L^2} \|\nabla u_t\|_{L^2} \leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \psi^\alpha, \tag{3.56}$$

$$R_5 \leq C \|\Delta \Phi\|_{L^\infty} \|\eta\|_{L^2} \|\nabla u_t\|_{L^2} \leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C, \tag{3.57}$$

$$\begin{aligned}
R_6 &\leq C \int \bar{x}^{\frac{a}{2}} |\nabla \eta| \bar{x}^{-\frac{a}{2}} |u_t| |\Delta \Phi| dx \\
&\leq C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^4} \|u_t \bar{x}^{-\frac{a}{2}}\|_{L^4} \|\Delta \Phi\|_{L^2} \\
&\leq C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^{\frac{1}{2}} \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^{\frac{1}{2}} (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2} \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2} + C \|\sqrt{\rho} u_t\|_{L^2}^2,
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
R_7 &\leq C \int \bar{x}^{\frac{a}{2}} \eta \bar{x}^{-\frac{a}{4}} |u| |\Delta \Phi| \bar{x}^{-\frac{a}{4}} |u_t| dx \\
&\leq C \|\Delta \Phi\|_{L^\infty} \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2} \|u \bar{x}^{-\frac{a}{4}}\|_{L^4} \|u_t \bar{x}^{-\frac{a}{4}}\|_{L^4} \\
&\leq C \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2} (\|\sqrt{\rho} u\|_{L^2} + \|\nabla u\|_{L^2}) (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \psi^\alpha \|\sqrt{\rho} u_t\|_{L^2}^2 + C \psi^\alpha,
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
R_8 &\leq C \int \bar{x}^{\frac{a}{2}} \eta \bar{x}^{-\frac{a}{2}} |u| |\nabla \Phi| |\nabla u_t| dx \\
&\leq C \|\nabla \Phi\|_{L^\infty} \|\bar{x}^{\frac{a}{2}} \eta\|_{L^4} \|u \bar{x}^{-\frac{a}{2}}\|_{L^4} \|\nabla u_t\|_{L^2} \\
&\leq C \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2}^{\frac{1}{2}} \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^{\frac{1}{2}} (\|\sqrt{\rho} u\|_{L^2} + \|\nabla u\|_{L^2}) \|\nabla u_t\|_{L^2} \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + C \psi^\alpha,
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
R_9 &\leq C \int \bar{x}^{\frac{a}{2}} \eta |\nabla \Phi| |\Delta \Phi| \bar{x}^{-\frac{a}{2}} |u_t| dx \\
&\leq C \|\Delta \Phi\|_{L^\infty} \|\nabla \Phi\|_{L^2} \|\bar{x}^{\frac{a}{2}} \eta\|_{L^4} \|u_t \bar{x}^{-\frac{a}{2}}\|_{L^4} \\
&\leq C \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2}^{\frac{1}{2}} \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^{\frac{1}{2}} (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}) \\
&\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + C \psi^\alpha \|\sqrt{\rho} u_t\|_{L^2}^2 + C \psi^\alpha,
\end{aligned}$$

$$R_{10} \leq C \|\nabla \Phi\|_{L^\infty}^2 \|\eta\|_{L^2} \|\nabla u_t\|_{L^2} \leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C, \tag{3.61}$$

$$\begin{aligned}
R_{11} &\leq C \int \bar{x}^{\frac{a}{2}} |\nabla \rho| \bar{x}^{-\frac{a}{4}} |u| |\nabla \Phi| \bar{x}^{-\frac{a}{4}} |u_t| dx \\
&\leq C \|\nabla \Phi\|_{L^\infty} \|\bar{x}^{\frac{a}{2}} \nabla \rho\|_{L^2} \|u \bar{x}^{-\frac{a}{4}}\|_{L^4} \|u_t \bar{x}^{-\frac{a}{4}}\|_{L^4} \\
&\leq C \psi^\alpha (\|\sqrt{\rho} u\|_{L^2} + \|\nabla u\|_{L^2}) (\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2})
\end{aligned}$$

$$\leq \frac{1}{12} \|\nabla u_t\|_{L^2}^2 + C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha, \quad (3.62)$$

$$\begin{aligned} R_{12} &\leq C \int \rho |\nabla \Phi| |u_t| |\nabla u| dx \\ &\leq C \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\nabla \Phi\|_{L^\infty} \|\sqrt{\rho}u_t\|_{L^2} \|\nabla u\|_{L^2} \\ &\leq C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha. \end{aligned} \quad (3.63)$$

Substituting (3.53)-(3.63) into (3.50), and we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int ((2\mu + \lambda)(\operatorname{div} u_t)^2 + \mu \omega_t^2 + R^{-1}|u_t|^2) dx \\ &\leq C\psi^\alpha (1 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) + C\psi^\alpha \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + \frac{1}{2} \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2 \\ &\leq \frac{1}{2} \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2 + C\psi^\alpha \|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + C\|\eta_t\|_{L^2}^2 + C\psi^\alpha, \end{aligned} \quad (3.64)$$

where in the last inequality we have used (3.44).

Next, we should estimate $\|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2$. Indeed, multiplying (2.2)₃ by $\Delta \eta \bar{x}^a$, integrating the result equation by parts yields that

$$\begin{aligned} &\frac{1}{2} \left(\int |\nabla \eta|^2 \bar{x}^a dx \right)_t + \int |\Delta \eta|^2 \bar{x}^a dx \\ &= - \int \eta_t \cdot \nabla \eta \nabla \bar{x}^a dx + \int \nabla \cdot (\eta u - \eta \nabla \Phi) \cdot \Delta \eta \bar{x}^a dx \\ &= - \int (\Delta \eta - \nabla \cdot (\eta u - \eta \nabla \Phi)) \cdot \nabla \eta \nabla \bar{x}^a dx + \int \nabla \cdot (\eta u - \eta \nabla \Phi) \cdot \Delta \eta \bar{x}^a dx \\ &\leq C \int |\nabla \eta| |\Delta \eta| |\nabla \bar{x}^a| dx + C \int |\nabla \eta|^2 |u| |\nabla \bar{x}^a| dx + C \int \eta |\nabla \eta| |\nabla u| |\nabla \bar{x}^a| dx \\ &\quad + C \int |\nabla \eta|^2 |\nabla \Phi| |\nabla \bar{x}^a| dx + C \int \eta |\nabla \eta| |\Delta \Phi| |\nabla \bar{x}^a| dx + C \int |\nabla u| |\nabla \eta|^2 \bar{x}^a dx \\ &\quad + C \int \eta |\nabla u| |\Delta \eta| \bar{x}^a dx + C \int |\nabla \eta| |\nabla \Phi| |\Delta \eta| \bar{x}^a dx + C \int \eta |\Delta \Phi| |\Delta \eta| \bar{x}^a dx \\ &\triangleq \sum_{i=1}^9 S_i. \end{aligned} \quad (3.65)$$

Using the Gagliardo-Nirenberg inequality, (3.15), (??), (3.4) and (3.5), we get

$$\begin{aligned} S_1 &\leq C \int \bar{x}^{\frac{a}{2}} |\nabla \eta| \bar{x}^{\frac{a}{2}} |\Delta \eta| \bar{x}^{-1} \log^{1+\sigma_0}(e + |x|^2) dx \\ &\leq \varepsilon \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2 + C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2, \end{aligned} \quad (3.66)$$

$$\begin{aligned} S_2 &\leq C \int |\nabla \eta|^{\frac{2a-1}{a}} \bar{x}^{\frac{2a-1}{2}} |\nabla \eta|^{\frac{1}{a}} |u| \bar{x}^{-\frac{1}{4}} \bar{x}^{-\frac{1}{4}} \log^{1+\sigma_0}(e + |x|^2) dx \\ &\leq C \|\bar{x}^{\frac{2a-1}{2}} |\nabla \eta|^{\frac{2a-1}{a}}\|_{L^{\frac{2a}{2a-1}}} \|u \bar{x}^{-\frac{1}{4}}\|_{L^{4a}} \|\nabla \eta|^{\frac{1}{a}}\|_{L^{4a}} \\ &\leq C\psi^\alpha \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + \varepsilon \|\bar{x}^{\frac{a}{2}} \Delta \eta\|_{L^2}^2, \end{aligned} \quad (3.67)$$

$$S_3 \leq C \int \bar{x}^{\frac{a}{2}} \eta |\nabla u| \bar{x}^{\frac{a}{2}} |\nabla \eta| \bar{x}^{-1} \log^{1+\sigma_0}(e + |x|^2) dx$$

$$\begin{aligned}
&\leq C\|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^4}\|\nabla u\|_{L^4}\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2} \\
&\leq C\|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^4}^4 + C\|\nabla u\|_{L^4}^4 + C\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 \\
&\leq C\|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^2}^2(\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + \|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^2}^2) + C\|\nabla u\|_{L^4}^4 + C\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 \\
&\leq C\psi^\alpha(\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2),
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
S_4 &\leq C \int |\nabla\eta|^2 |\nabla\Phi| \bar{x}^a \bar{x}^{-1} \log^{1+\sigma_0}(e + |x|^2) dx \\
&\leq C\|\nabla\Phi\|_{L^\infty} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2,
\end{aligned} \tag{3.69}$$

$$\begin{aligned}
S_5 &\leq C \int \bar{x}^{\frac{\alpha}{2}}\eta |\Delta\Phi| \bar{x}^{\frac{\alpha}{2}} |\nabla\eta| \bar{x}^{-1} \log^{1+\sigma_0}(e + |x|^2) dx \\
&\leq C\|\Delta\Phi\|_{L^\infty} \|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^2}^2 + C\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
S_6 &\leq C\|\nabla u\|_{L^2} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^4}^2 \\
&\leq C\|\nabla u\|_{L^2} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2} (\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2} + \|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}) \\
&\leq \varepsilon\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2} + C(\varepsilon)\psi^\alpha\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2},
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
S_7 &\leq C\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2} \|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^4} \|\nabla u\|_{L^4} \\
&\leq \varepsilon\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 + C\|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^2} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \\
&\leq \varepsilon\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 + C\psi^{-1}\|\nabla^2 u\|_{L^2}^2 + C\psi^\alpha\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2,
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
S_8 + S_9 &\leq C(\|\nabla\Phi\|_{L^\infty} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2} + \|\Delta\Phi\|_{L^\infty} \|\bar{x}^{\frac{\alpha}{2}}\eta\|_{L^2}) \|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2} \\
&\leq \varepsilon\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 + C\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + C\psi^\alpha.
\end{aligned} \tag{3.73}$$

Substituting (3.66)-(3.73) into (3.65) and choosing ε suitably small lead to

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + \|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 \\
&\leq \varepsilon\|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 + C\psi^\alpha\|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + C\psi^{-1}\|\nabla^2 u\|_{L^2}^2 + C\psi^\alpha.
\end{aligned} \tag{3.74}$$

Thus, multiplied (3.74) by s , together with Gronwall's inequality, we get

$$\sup_{0 \leq s \leq t} s \|\bar{x}^{\frac{\alpha}{2}}\nabla\eta\|_{L^2}^2 + \int_0^t s \|\bar{x}^{\frac{\alpha}{2}}\Delta\eta\|_{L^2}^2 ds \leq C \exp \left\{ \int_0^t \psi^\alpha ds \right\}, \tag{3.75}$$

due to (3.44) and (3.12).

Now, multiplying (3.64) by t , we obtain (3.48) using Gronwall's inequality and (3.75). The proof of Lemma 3.3 is completed.

Lemma 3.4 *Let (ρ, u, η) and T_1 be as in Lemma 3.2. Then, for all $t \in (0, T_1]$,*

$$\sup_{0 \leq s \leq t} \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} \leq \exp \left\{ C \exp \left\{ \int_0^t \psi^\alpha ds \right\} \right\}. \tag{3.76}$$

Proof. Notice that following the framework of Lemma 3.4 in [21] for proving an estimate similar to (3.76), it suffices to verify the following estimate:

$$\int_0^t (\|\nabla^2 u\|_{L^2 \cap L^q}^{\frac{q+1}{q}} + s \|\nabla^2 u\|_{L^2 \cap L^q}^2) ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \tag{3.77}$$

In fact, on the one hand, it follows from (3.44), (3.48) and (3.38) that

$$\begin{aligned}
& \int_0^t (\|\nabla^2 u\|_{L^2}^{\frac{5}{3}} + s\|\nabla^2 u\|_{L^2}^2) ds \\
& \leq C \int_0^t (\psi^\alpha + \|\sqrt{\rho} u_t\|_{L^2}^2) ds + C \sup_{0 \leq s \leq t} (s\|\sqrt{\rho} u_t\|_{L^2}^2) \int_0^t \psi ds \\
& \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}.
\end{aligned} \tag{3.78}$$

On the other hand, choosing $p = q$ in (3.43), using the Gagliardo-Nirenberg inequality gives

$$\begin{aligned}
\|\nabla^2 u\|_{L^q} & \leq C(\|\rho \dot{u}\|_{L^p} + \|(\eta + \beta\rho)\nabla\Phi\|_{L^p} + \|\nabla p_F\|_{L^p} + \|\nabla\eta\|_{L^p}) \\
& \leq C(\|\rho u_t\|_{L^q} + \|\rho u\|_{L^{2q}}\|\nabla u\|_{L^{2q}} + \|\eta\|_{L^p} + \psi^\alpha + \|\nabla\eta\|_{L^p}) \\
& \leq C\|\rho u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\rho u\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + C\psi^\alpha(1 + \|\nabla^2 u\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \eta\|_{L^2}^{1-\frac{1}{q}}) \\
& \leq C\psi^\alpha(\|\sqrt{\rho} u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}} \|\nabla u_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + \|\sqrt{\rho} u_t\|_{L^2}) + C\psi^\alpha(1 + \|\nabla^2 u\|_{L^2}^{1-\frac{1}{q}} + \|\nabla^2 \eta\|_{L^2}^{1-\frac{1}{q}}).
\end{aligned} \tag{3.79}$$

This combined with (3.78) (3.48), (3.4), and (3.5)

$$\begin{aligned}
\int_0^t \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} ds & \leq C \int_0^t \psi^\alpha s^{-\frac{q+1}{2q}} (s\|\sqrt{\rho} u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}} (s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
& \quad + C \int_0^t \psi^\alpha \|\sqrt{\rho} u_t\|_{L^2}^{\frac{q+1}{q}} ds + C \int_0^t \left(1 + \|\nabla^2 u\|_{L^2}^{\frac{q^2-1}{q^2}} + \|\nabla^2 \eta\|_{L^2}^{\frac{q^2-1}{q^2}}\right) ds \\
& \leq C \sup_{0 \leq s \leq t} (s\|\sqrt{\rho} u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}} \int_0^t \psi^\alpha s^{-\frac{q+1}{2q}} (s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}} ds \\
& \quad + C \int_0^t \left(\psi^\alpha + \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^{\frac{5}{3}} + \|\nabla^2 \eta\|_{L^2}^2\right) ds \\
& \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \left[1 + \int_0^t \left(\psi^\alpha + s^{-\frac{q^3+q^2-2q-1}{q^3+q^2-2q}} + s\|\nabla u_t\|_{L^2}^2\right) ds\right] \\
& \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\},
\end{aligned} \tag{3.80}$$

and that

$$\int_0^t s\|\nabla^2 u\|_{L^q}^2 ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \tag{3.81}$$

One thus obtains (3.77) from (3.78)- and completes the proof of lemma 3.4.

Now, proposition 3.1 is a direct consequence of lemmas 3.1-3.4.

Proof of proposition 3.1. It follows from (3.76), (3.4), (3.5), and (3.11) and that

$$\psi(t) \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\}.$$

Standard arguments thus yield that for $M \triangleq e^{Ce}$ and $T_0 \triangleq \min\{T_1, (CM^\alpha)^{-1}\}$,

$$\sup_{0 \leq t \leq T_0} \psi(t) \leq M,$$

which together with (3.44), (3.77) and (3.12). The proof of Proposition 3.1 is thus completed.

Lemma 3.5 *Let (ρ, u, η) be a smooth solution to the initial-boundary-value problem (2.2), and T_0 is obtained in proposition (3.1), then we have*

$$\sup_{0 \leq s \leq T_0} (s \|\eta_t\|_{L^2}^2 + s \|\Delta \eta\|_{L^2}^2) + \int_0^{T_0} s \|\nabla \eta_t\|_{L^2}^2 ds \leq C. \quad (3.82)$$

Proof. Differentiating (2.2)₃ with respect to t shows

$$\eta_{tt} + \nabla \cdot (\eta_t u + \eta u_t - \eta_t \nabla \Phi) - \Delta \eta_t = 0, \quad (3.83)$$

Multiplying (3.83) by η_t and then integrating equation over B_R , integrating by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\eta_t|^2 dx + \int |\nabla \eta_t|^2 dx \\ &= \int (\eta_t u + \eta u_t - \eta_t \nabla \Phi) \cdot \nabla \eta_t dx \\ &\leq C \int |\eta_t|^2 |\nabla u| dx + C \int \eta |u| |\Delta \Phi| |\nabla \eta_t| dx \\ &\quad + C \int \eta |u_t| |\nabla \eta_t| dx + C \int |\eta_t| |\nabla \Phi| |\nabla \eta_t| dx \triangleq \sum_{i=1}^4 K_i. \end{aligned} \quad (3.84)$$

Using the Hölder's inequality, Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} K_1 &\leq C \|\eta_t\|_{L^4}^2 \|\nabla u\|_{L^2} \\ &\leq C \|\eta_t\|_{L^2} \|\nabla \eta_t\|_{L^2} \|\nabla u\|_{L^2} \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C(\varepsilon) \|\eta_t\|_{L^2}, \end{aligned} \quad (3.85)$$

$$\begin{aligned} K_2 &\leq \int \eta \bar{x}^{\frac{a}{2}} |u| \bar{x}^{\frac{a}{2}} |\nabla \Phi| |\nabla \eta_t| dx \\ &\leq C \|\nabla \Phi\|_{L^\infty} \|\nabla \eta_t\|_{L^2} \|\bar{x}^{\frac{a}{2}} \eta\|_{L^4} \|u \bar{x}^{-\frac{a}{2}}\|_{L^4} \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}, \end{aligned} \quad (3.86)$$

$$\begin{aligned} K_3 &\leq C \|\nabla \eta_t\|_{L^2} \|\eta |u_t|\|_{L^2} \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C \int \eta \bar{x}^{\frac{a}{2}} \eta |u_t|^2 \bar{x}^{-\frac{a}{2}} dx \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C \|\eta\|_{L^4} \|u_t \bar{x}^{-\frac{a}{2}}\|_{L^8}^2 \|\bar{x}^{\frac{a}{2}} \eta\|_{L^2} \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2, \end{aligned} \quad (3.87)$$

$$\begin{aligned} K_4 &\leq C \|\nabla \Phi\|_{L^\infty} \|\eta_t\|_{L^2} \|\nabla \eta_t\|_{L^2} \\ &\leq \varepsilon \|\nabla \eta_t\|_{L^2}^2 + C(\varepsilon) \|\eta_t\|_{L^2}^2. \end{aligned} \quad (3.88)$$

Now, putting (3.85) and (3.88) into (3.84), and multiplying the resulting inequality by s , we have after choosing ε suitably small that

$$\begin{aligned} & \frac{d}{dt} (s \|\eta_t\|_{L^2}^2) + s \|\nabla \eta_t\|_{L^2}^2 \\ &\leq C (s \|\eta_t\|_{L^2}^2) + C (\|\eta_t\|_{L^2}^2 + s \|\bar{x}^{\frac{a}{2}} \nabla \eta\|_{L^2}^2 + s \|\nabla u_t\|_{L^2}^2 + s \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2), \end{aligned} \quad (3.89)$$

which together with Gronwall's inequality and (3.38) yields that

$$\sup_{0 \leq s \leq T_0} (s \|\eta_t\|_{L^2}^2 + s \|\Delta \eta\|_{L^2}^2) + \int_0^{T_0} s \|\nabla \eta_t\|_{L^2}^2 ds \leq C. \quad (3.90)$$

The proof of lemma 3.5 is completed.

4 Proofs of theorems 1.1

Let (ρ_0, u_0, η_0) be as in Theorem 1.1. For simplicity, assume that

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \quad (4.1)$$

We construct $\rho_0^R = \hat{\rho}_0^R + R^{-1}e^{-|x|^2}$ where $0 \leq \hat{\rho}_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies that

$$\int_{B_{N_0}} \hat{\rho}_0^R dx \geq \frac{1}{2}, \quad (4.2)$$

and that

$$\bar{x}^a \hat{\rho}_0^R \rightarrow \bar{x}^a \rho_0, \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2) \quad \text{as } R \rightarrow \infty. \quad (4.3)$$

Notice that $\bar{x}^{\frac{a}{2}} \eta_0 \in L^2(\mathbb{R}^2)$ and $\bar{x}^{\frac{a}{2}} \nabla \eta_0 \in L^2(\mathbb{R}^2)$, choosing $\eta_0^R \in C_0^\infty(B_R)$ such that

$$\bar{x}^{\frac{a}{2}} \eta_0^R \rightarrow \bar{x}^{\frac{a}{2}} \eta_0, \quad \nabla \eta_0^R \rightarrow \nabla \eta_0 \quad \text{in } L^2(\mathbb{R}^2), \quad \text{as } R \rightarrow \infty. \quad (4.4)$$

Since $\nabla u_0 \in L^2(\mathbb{R}^2)$, choosing $v_i^R \in C_0^\infty(B_R)$ ($i = 1, 2$) such that

$$\lim_{R \rightarrow \infty} \|v_i^R - \partial_i u_0\|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2, \quad (4.5)$$

and let smooth u_0^R uniquely solve

$$\begin{cases} -\Delta u_0^R + R^{-1} u_0^R = -\rho_0^R u_0^R + \sqrt{\rho_0^R} h^R - \partial_i v_i^R & \text{in } B_R, \\ u_0^R = 0 & \text{on } \partial B_R, \end{cases} \quad (4.6)$$

where $h^R = (\sqrt{\rho_0} w_0^R) * j_{1/R}$ with the standard mollifying kernel j_δ , $\delta > 0$. Extend u_0^R to \mathbb{R}^2 by defining 0 outside B_R , and denote $w_0^R \triangleq u_0^R \varphi_R$. By the same arguments as those for the proof of Theorem 1.1 in [21], we obtained that

$$\lim_{R \rightarrow \infty} \left(\|\nabla(w_0^R - u_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R} w_0^R - \sqrt{\rho_0} u_0\|_{L^2(\mathbb{R}^2)} \right) = 0, \quad (4.7)$$

where

$$0 \leq \varphi_R \leq 1, \quad \varphi_R(x) = 1, \quad \text{if } |x| \leq R/2, \quad |\nabla^k \varphi_R| \leq CR^{-k} (k = 1, 2). \quad (4.8)$$

Then, in terms of lemma 2.1, the initial-boundary value problem (2.2) with the initial data $(\rho_0^R, u_0^R, \eta_0^R)$ has a classical solution (ρ^R, u^R, η^R) on $B_R \times [0, T_R]$. Moreover, proposition 3.1 show that exists a T_0 independent of R such that (3.3) and (3.82) hold for (ρ^R, u^R, η^R) . By (3.3), (3.11), (4.4), (4.7), and (3.82), after taking a subsequence, (ρ^R, u^R, η^R) locally and weakly (in the corresponding spaces) converges to a strong solution (ρ, u, η) of (1.1)-(1.5) on $\mathbb{R}^2 \times (0, T_0]$ satisfying (1.7) and (1.8). The proof of the existence part of theorem 1.1 is completed.

Next prove the uniqueness of the strong solutions. Take two strong solutions (ρ_i, u_i, η_i) ($i = 1, 2$) sharing the same initial data with (1.7) and (1.8), and let $\bar{\rho} = \rho_2 - \rho_1$, $\bar{u} = u_2 - u_1$, $\bar{\eta} = \eta_2 - \eta_1$. Then,

$$\begin{cases} \bar{\rho}_t + (u_2 \cdot \nabla) \bar{\rho} + \bar{u} \cdot \nabla \rho_1 + \bar{\rho} \operatorname{div} u_2 + \rho_1 \operatorname{div} \bar{u} = 0, \\ \rho_1 \bar{u}_t + \rho_1 u_1 \cdot \nabla \bar{u} + \nabla(p_F(\rho_2) - p_F(\rho_1)) + \nabla(\eta_2 - \eta_1) \\ \quad = \mu \Delta \bar{u} + (\mu + \lambda) \nabla \operatorname{div} \bar{u} - \bar{\rho}(u_{2t} + u_2 \cdot \nabla u_2) - \rho_1 \bar{u} \cdot \nabla u_2 - (\bar{\eta} + \beta \bar{\rho}) \nabla \Phi, \\ \bar{\eta}_t + \nabla \cdot (\bar{\eta} u_2 - \bar{\eta} \nabla \Phi) + \nabla \cdot (\eta_2 \bar{u}) - \Delta \bar{\eta} = 0. \end{cases} \quad (4.9)$$

for $(x, t) \in \mathbb{R}^2 \times (0, T_0]$ with

$$\bar{\rho}(x, 0) = \bar{u}(x, 0) = \bar{\eta}(x, 0) = 0, \quad x \in \mathbb{R}^2. \quad (4.10)$$

Firstly, multiply (4.9)₁ by $2\bar{\rho}\bar{x}^{2r}$ and integrate by parts. Similar to the inequality (5.32) in [21], we get that

$$\|\bar{\rho}\bar{x}^r\|_{L^2} \leq C \int_0^t (\|\nabla \bar{u}\|_{L^2} + \|\sqrt{\rho_1} \bar{u}\|_{L^2}) ds, \quad t \in (0, T_0], \quad (4.11)$$

where $r \in (1, \tilde{a})$ with $\tilde{a} = \min\{a, 2\}$.

Secondly, multiplying (4.9)₂ by \bar{u} and integrating by parts lead to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho_1 |\bar{u}|^2 dx + \int (2\mu + \lambda) |\operatorname{div} \bar{u}|^2 + \mu |\omega|^2 dx \\ &= - \int \bar{\rho}(u_{2t} + u_2 \cdot \nabla u_2) \cdot \bar{u} dx - \int \rho_1 \bar{u} \cdot \nabla u_2 \cdot \bar{u} dx \\ & \quad + \int (p_F(\rho_2) - p_F(\rho_1)) \operatorname{div} \bar{u} dx + \int \bar{\eta} \operatorname{div} \bar{u} dx + \int (\bar{\eta} + \beta \bar{\rho}) \nabla \Phi \bar{u} dx \\ &\leq C \|\nabla u_2\|_{L^\infty} \int \rho_1 |\bar{u}|^2 dx + C \int |\bar{\rho}| |\bar{u}| (|u_{2t}| + |u_2| |\nabla u_2|) dx \\ & \quad + C \|p_F(\rho_2) - p_F(\rho_1)\|_{L^2} \|\operatorname{div} \bar{u}\|_{L^2} + C \|\bar{\eta}\|_{L^2} \|\operatorname{div} \bar{u}\|_{L^2} \\ & \quad + C \int \bar{\eta} \bar{x}^{\frac{r}{2}} |\bar{u}| \bar{x}^{-\frac{r}{2}} |\nabla \Phi| dx + C \int \bar{\rho} \bar{x}^{\frac{r}{2}} |\bar{u}| \bar{x}^{-\frac{r}{2}} |\nabla \Phi| dx \\ &\triangleq C \|\nabla u_2\|_{L^\infty} \int \rho_1 |\bar{u}|^2 dx + \sum_{i=1}^5 Q_i. \end{aligned} \quad (4.12)$$

Just like (4.11), it has been obtained via (5.33) and (5.36) in [21] that

$$\begin{aligned} Q_1 + Q_2 &\leq C(\varepsilon) (1 + t \|\nabla u_{2t}\|_{L^2}^2 + t \|\nabla^2 u_2\|_{L^q}^2) \int_0^t (\|\nabla \bar{u}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2) ds \\ & \quad + \varepsilon (\|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2). \end{aligned} \quad (4.13)$$

With the Cauchy inequality and (3.3), (3.14), and (3.82), we have

$$\begin{aligned} \sum_{i=3}^5 Q_i &\leq \varepsilon \|\nabla \bar{u}\|_{L^2}^2 + C(\varepsilon) \|\bar{\eta}\|_{L^2}^2 + C \|\nabla \Phi\|_{L^4} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} \|\bar{u} \bar{x}^{-\frac{r}{2}}\|_{L^4} \\ & \quad + C \|\nabla \Phi\|_{L^4} \|\bar{x}^{\frac{r}{2}} \bar{\rho}\|_{L^2} \|\bar{u} \bar{x}^{-\frac{r}{2}}\|_{L^4} \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon \|\nabla \bar{u}\|_{L^2}^2 + C(\varepsilon) \|\bar{\eta}\|_{L^2}^2 + C \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} (\|\sqrt{\rho_1} \bar{u}\|_{L^2} + \|\nabla \bar{u}\|_{L^2}) \\
&\quad + C(\|\sqrt{\rho} \bar{u}\|_{L^2} + \|\nabla \bar{u}\|_{L^2}) \int_0^t (\|\nabla \bar{u}\|_{L^2} + \|\sqrt{\rho_1} \bar{u}\|_{L^2}) ds \\
&\leq \varepsilon (\|\nabla \bar{u}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2) + C(\varepsilon) \|\bar{\eta}\|_{L^2}^2 + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2}^2 \\
&\quad + \int_0^t (\|\nabla \bar{u}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2) ds.
\end{aligned} \tag{4.14}$$

It remains to estimate the $\|\bar{\eta}\|_{L^2}$ and $\|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2}$. In fact, multiplying (1.1)₃ by $\bar{\eta}$ and integrating

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |\bar{\eta}|^2 dx + \|\nabla \bar{\eta}\|_{L^2}^2 \\
&= - \int \bar{\eta} u_2 \cdot \nabla \bar{\eta} dx - \int \bar{\eta}^2 \operatorname{div} u_2 dx + \int \bar{\eta} \nabla \bar{\eta} \cdot \nabla \Phi dx \\
&\quad + \int \bar{\eta}^2 \Delta \Phi dx - \int \bar{\eta} \bar{u} \cdot \nabla \eta_2 dx - \int \eta_2 \operatorname{div} \bar{u} \bar{\eta} dx \\
&\triangleq \sum_{i=1}^6 II_i.
\end{aligned} \tag{4.15}$$

Using the Hölder's inequality, Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
II_1 &\leq C \int \bar{x}^{\frac{r}{2}} \bar{\eta} \bar{x}^{-\frac{r}{2}} |u_2| |\nabla \bar{\eta}| dx \\
&\leq C \|\nabla \bar{\eta}\|_{L^2} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^4} \|u_2 \bar{x}^{-\frac{r}{2}}\|_{L^4} \\
&\leq \varepsilon \|\nabla \bar{\eta}\|_{L^2}^2 + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2}^2 + \delta \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2}^2,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\sum_{i=2}^4 II_i &\leq C \|\nabla u_2\|_{L^2} \|\bar{\eta}\|_{L^4}^2 + C \|\Delta \Phi\|_{L^\infty} \|\eta\|_{L^2}^2 \\
&\quad + \|\nabla \Phi\|_{L^\infty} \|\bar{\eta}\|_{L^2} \|\nabla \bar{\eta}\|_{L^2} \\
&\leq \varepsilon \|\nabla \bar{\eta}\|_{L^2}^2 + C(\varepsilon) \|\bar{\eta}\|_{L^2}^2 + C,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
II_5 &\leq C \int \bar{x}^{\frac{r}{2}} \bar{\eta} \bar{x}^{-\frac{r}{2}} |\bar{u}| |\nabla \eta_2| dx \\
&\leq C \|\nabla \eta_2\|_{L^2} \|\bar{u} \bar{x}^{-\frac{r}{2}}\|_{L^4} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^4} \\
&\leq \varepsilon (\|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2) + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2}^2 + \varepsilon \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2}^2,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
II_6 &\leq C \|\nabla \bar{u}\|_{L^2} \|\eta_2\|_{L^4} \|\bar{\eta}\|_{L^4} \\
&\leq \varepsilon \|\nabla \bar{u}\|_{L^2}^2 + \delta \|\nabla \bar{\eta}\|_{L^2}^2 + C(\varepsilon, \delta) \|\bar{\eta}\|_{L^2}^2.
\end{aligned} \tag{4.19}$$

Moreover, multiplying (1.1)₃ by $\bar{x}^r \bar{\eta}$ and integrating by parts yield

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int |\bar{\eta}|^2 \bar{x}^r dx + \int |\nabla \bar{\eta}|^2 \bar{x}^r dx \\
&= \int \bar{\eta}^2 \Delta \bar{x}^r dx - \int u_2 \cdot \nabla \bar{\eta} \bar{\eta} \bar{x}^r dx - \int \bar{\eta} \operatorname{div} u_2 \bar{\eta} \bar{x}^r dx \\
&\quad + \int \nabla \bar{\eta} \nabla \cdot \Phi \bar{\eta} \bar{x}^r dx + \int \bar{\eta}^2 \bar{x}^r \Delta \Phi dx + \int \bar{u} \eta_2 \nabla \bar{\eta} \bar{x}^r dx
\end{aligned}$$

$$+ \int \bar{u} \eta_2 \bar{\eta} \nabla \bar{x}^r dx \triangleq \sum_{i=1}^7 III_i. \quad (4.20)$$

For the term $III_i (i = 1, \dots, 7)$ on the right hand side of (4.20), we get that

$$III_1 \leq C \int |\bar{\eta}|^2 \bar{x}^r \bar{x}^{-2} \log^{2(1+\sigma_0)}(e + |x|^2) dx \leq C \int |\bar{\eta}|^2 \bar{x}^r dx, \quad (4.21)$$

$$\begin{aligned} III_2 + III_3 &= -2 \int \bar{\eta}^2 \operatorname{div} u_2 \bar{x}^r dx - \int \bar{\eta}^2 u_2 \nabla \bar{x}^r dx \\ &\leq C \|\nabla u_2\|_{L^2} \|\bar{x}^{\frac{a}{2}} \bar{\eta}\|_{L^4}^2 + C \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^4} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} \|u_2 \bar{x}^{-\frac{3}{4}}\|_{L^4} \\ &\leq \varepsilon \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2}^2 + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} + C, \end{aligned} \quad (4.22)$$

$$\begin{aligned} III_4 + III_5 &\leq \|\nabla \Phi\|_{L^\infty} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2} + C \|\Delta \Phi\|_{L^\infty} \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} \\ &\leq \varepsilon \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2}^2 + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} + C, \end{aligned} \quad (4.23)$$

$$\begin{aligned} III_6 &\leq \int \bar{u} \bar{x}^{-\frac{b}{2}} \eta_2 \bar{x}^{\frac{b+r}{2}} \nabla \bar{\eta} \bar{x}^{\frac{r}{2}} dx \\ &\leq \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2} \|\bar{u} \bar{x}^{-\frac{b}{2}}\|_{L^4} \|\bar{x}^{\frac{a}{2}} \eta_2\|_{L^4} \\ &\leq \varepsilon \|\bar{x}^{\frac{r}{2}} \nabla \bar{\eta}\|_{L^2}^2 + \varepsilon (\|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2) + C, \end{aligned} \quad (4.24)$$

$$\begin{aligned} III_7 &\leq C \int \bar{u} \bar{x}^{-\frac{3}{4}} \eta_2 \bar{x}^{\frac{r}{2}} \bar{\eta} \bar{x}^{\frac{r}{2}} \bar{x}^{-\frac{1}{4}} \log^{1+\sigma_0}(e + |x|^2) dx \\ &\leq C \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} \|\bar{u} \bar{x}^{-\frac{3}{4}}\|_{L^4} \|\bar{x}^{\frac{r}{2}} \eta_2\|_{L^4} \\ &\leq \varepsilon (\|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2) + C(\varepsilon) \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2}^2, \end{aligned} \quad (4.25)$$

where $b + r < \tilde{a}$.

Denoting

$$G(t) \triangleq \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\bar{\eta}\|_{L^2} + \|\bar{x}^{\frac{r}{2}} \bar{\eta}\|_{L^2} + \int_0^t (\|\nabla \bar{u}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2) ds, \quad (4.26)$$

with all these estimates (4.13)-(4.25), choosing ε, δ suitably small lead to

$$G'(t) \leq C(1 + \|\nabla u_2\|_{L^\infty} + t \|\nabla u_{2t}\|_{L^2}^2 + t \|\nabla^2 u_2\|_{L^q}^2) G(t), \quad (4.27)$$

which together with Gronwall's inequality, and (1.7) yields $G(t) = 0$. Hence, $\bar{u}(x, t) = 0$ and $\bar{\eta}(x, t) = 0$ for almost everywhere $(x, t) \in \mathbb{R}^2 \times (0, T_0)$. Then, one can deduce from (4.11) that $\bar{\rho} = 0$ for almost everywhere $(x, t) \in \mathbb{R}^2 \times (0, T_0)$. The proof of theorem 1.1 is completed.

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